

Zero energy correction method for non-Hermitian Harmonic oscillator with simultaneous transformation of co-ordinate and momentum:Wave function analysis under Iso-spectral condition.

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We present a complete analysis on energy and wavefunction of Harmonic oscillator with simultaneous non-hermitian transformation of co-ordinate ($x \rightarrow \frac{(x+i\lambda p)}{\sqrt{(1+\beta\lambda)}}$) and momentum ($p \rightarrow \frac{(p+i\beta x)}{\sqrt{(1+\beta\lambda)}}$) using perturbation theory under iso-spectral condition. Further we notice that two different frequency of oscillation (w_1, w_2) correspond to same energy eigenvalue, which can also be verified using Lie algebraic approach [Zhang et.al J.Math.Phys **56**, 072103 (2015)]. Interestingly wave function analysis using similarity transformation [F.M. Fernandez, Int. J. Theo. Phys. (2015)(in Press)] refers to a very special case.

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I.INTRODUCTION

In quantum mechanics, Harmonic oscillator plays a major role in understanding limitations of various approximation methods such as variational method, W.K.B. method, perturbation method etc. The basic advantage of this oscillator is that its energy levels and eigenfunctions are exactly known [1]. However, when the co-ordinate and momentum simultaneously under goes a non-Hermitian transformation

[2-4], its energy eigenvalue can be iso-spectral to original Harmonic oscillator. Under iso-spectral condition, the wavefunctions of the transformed Hamiltonian differ drastically from that of the original Hamiltonian. However, a complete picture on wavefunction is still a missing subject. Further, no such explicit calculations on wavefunctions are available at present. Hence the aim of this paper is to present a complete picture on wavefunction and energy under iso-spectral condition using perturbation theory.

II. Energy levels and Wavefunction of Simple Harmonic Oscillator (SHO)

Let's consider the Hamiltonian of SHO [1]

$$H_{HO} = \frac{p^2}{2} + \frac{x^2}{2} \quad (1)$$

whose exact energy eigenvalues is

$$E_n = (n + \frac{1}{2}) \quad (2)$$

and corresponding wavefunction is

$$\psi_n = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}} H_n(x) e^{-\frac{x^2}{2}} \quad (3)$$

where $H_n(x)$ is the Hermite polynomeal.

III. SHO under non-Hermitian transformation of co-ordinate (x) and momentum (p)

Let us consider the non-Hermitian transformation of x and p as [2-4]

$$x \rightarrow \frac{x + i\lambda p}{\sqrt{(1 + \beta\lambda)}} \quad (4)$$

and

$$p \rightarrow \frac{p + i\beta x}{\sqrt{(1 + \beta\lambda)}} \quad (5)$$

In this transformation we noticed that the transformed co-ordinate and momentum preserve the commutation relation i.e.

$$[x, p] = i \quad (6)$$

Now the new Hamiltonian with transformed x and p becomes non-Hermitian in nature and is

$$H = \frac{(p + i\beta x)^2}{2(1 + \lambda\beta)} + \frac{(x + i\lambda p)^2}{2(1 + \lambda\beta)} \quad (7)$$

IV. Second Quantization and Hamiltonian

In order to solve the above Hamiltonian (Eqn. (7)), we use the second quantization formalism as

$$x = \frac{(a + a^+)}{\sqrt{2\omega}} \quad (8)$$

and

$$p = i\sqrt{\frac{\omega}{2}}(a^+ - a) \quad (9)$$

where the creation operator, a^+ and annihilation operator a satisfy the commutation relation

$$[a, a^+] = 1 \quad (10)$$

and ω is an unknown parameter. The Hamiltonian in Eqn. (7) can be written as

$$H = H_D + H_N \quad (11)$$

where

$$H_D = [(1 - \lambda^2)\omega + \frac{(1 - \beta^2)}{\omega}] \frac{(2a^+a + 1)}{4(1 + \lambda\beta)} \quad (12)$$

and

$$H_N = U \frac{a^2}{4(1 + \lambda\beta)} + V \frac{(a^+)^2}{4(1 + \lambda\beta)} \quad (13)$$

$$V = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} - 2(\lambda + \beta)] \quad (14)$$

$$U = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} + 2(\lambda + \beta)] \quad (15)$$

V.(a). Zero Energy Correction Method (Case Study for $U=0$)

Now we solve the the eigenvalue relation:

$$H\Psi_n(x) = \epsilon_n\Psi_n(x) \quad (16)$$

using perturbation theory as follows. Here we express

$$\epsilon_n = \epsilon_n^{(0)} + \sum_{m=1}^k \epsilon_n^{(m)} \quad (17)$$

The zeroth order energy $\epsilon_n^{(0)}$ satisfies the the following eigenvalue relation

$$H_D|\psi_n\rangle = H_D|n\rangle = \epsilon_n^{(0)}|n\rangle \quad (18)$$

where $\psi_n^{(0)}$ is the zeroth order wave function and $\epsilon_n^{(m)}$ is the mth order perturbation correction.

$$\epsilon_n^{(0)} = \frac{(2n+1)}{4(1+\lambda\beta)} \left[(1-\lambda^2)\omega + \frac{(1-\beta^2)}{\omega} \right] \quad (19)$$

and

$$\sum_{m=1}^k \epsilon_n^{(m)} = \epsilon_n^{(1)} + \epsilon_n^{(2)} + \epsilon_n^{(3)} + \dots \quad (20)$$

The energy correction terms will give zero contribution if the parameter is determined from non-diagonal terms of H_N [2]

Let the coefficient of a^2 is zero [5] i.e.

$$U = [-\omega(1-\lambda^2) + \frac{(1-\beta^2)}{\omega} + 2(\lambda+\beta)] = 0 \quad (21)$$

which leads to(considering positive sign)

$$\omega = \omega_1 = \frac{(1+\beta)}{(1-\lambda)} \quad (22)$$

In this case,

$$\epsilon_n^{(0)} = (n + \frac{1}{2}) \quad (23)$$

Now the perturbation correction term is

$$H_N = V \frac{(a^+)^2}{4(1+\lambda\beta)} = -\frac{(\lambda+\beta)}{1+\lambda\beta} (a^+)^2 \quad (24)$$

In this case one will notice that

$$\langle n|H_N|n-2\rangle = V \frac{\sqrt{n(n-1)}}{4(1+\lambda\beta)} \quad (25)$$

$$\langle n-2|H_N|n\rangle = 0 \quad (26)$$

Hence it is easy to note that all orders of energy corrections will be zero. Let us consider explicitly corrections up to third order using standard perturbation series given in literature[1,2,6-8], which can be written as

$$\epsilon_n^{(1)} = \langle \psi_n|H_N|\psi_n\rangle = 0 \quad (27)$$

$$\epsilon_n^{(2)} = \sum_{k \neq n} \frac{\langle \psi_n|H_N|\psi_k\rangle \langle \psi_k|H_N|\psi_n\rangle}{(\epsilon_n^{(0)} - \epsilon_k^{(0)})} = \frac{\langle \psi_n|H_N|\psi_{n+2}\rangle \langle \psi_{n+2}|H_N|\psi_n\rangle}{(\epsilon_n^{(0)} - \epsilon_{n+2}^{(0)})} = 0 \quad (28)$$

$$\epsilon_n^{(3)} = \sum_{p,q} \frac{\langle \psi_n|H_N|\psi_p\rangle \langle \psi_p|H_N|\psi_q\rangle \langle \psi_q|H_N|\psi_n\rangle}{(\epsilon_n^{(0)} - \epsilon_p^{(0)})(\epsilon_n^{(0)} - \epsilon_q^{(0)})} = 0 \quad (29)$$

or

$$\epsilon_n^{(3)} = \frac{\langle \psi_n|H_N|\psi_{n+2}\rangle \langle \psi_{n+2}|H_N|\psi_{n+4}\rangle \langle \psi_{n+4}|H_N|\psi_n\rangle}{(\epsilon_n^{(0)} - \epsilon_{n+2}^{(0)})(\epsilon_n^{(0)} - \epsilon_{n+4}^{(0)})} = 0 \quad (30)$$

Here second order correction is zero due to $\langle \psi_n|H_N|\psi_{n+2}\rangle = \delta_{n,n+4}$ and third order correction is zero due to $\langle \psi_{n+4}|H_N|\psi_n\rangle = \delta_{n+4,n+2}$. Similarly one can notice all correction terms $\epsilon_n^{(m)}$ will be zero. Hence

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = (n + \frac{1}{2}) \quad (31)$$

$$|\psi_n\rangle = \left(\frac{\sqrt{\omega_1}}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}} H_n(\sqrt{\omega_1}x) e^{-\omega_1 \frac{x^2}{2}} \quad (32)$$

with

$$\langle \psi_n|\psi_n\rangle = 1 \quad (33)$$

Form Eq.(2) and Eq.(23), one can see that the total energy of the Harmonic oscillator and the Harmonic oscillator with non-hermitian transformation remains the same.

V.(b). Corresponding Wavefunction using Perturbation Theory

Here we find the wavefunction as

$$\Psi_n^{(k)} = |\psi_n\rangle + f_{\lambda,\beta} \frac{\sqrt{(n+2)!}}{2\sqrt{n!}} |\psi_{n+2}\rangle + (f_{\lambda,\beta})^2 \frac{\sqrt{(n+4)!}}{8\sqrt{n!}} |\psi_{n+4}\rangle + (f_{\lambda,\beta})^3 \frac{\sqrt{(n+6)!}}{48\sqrt{n!}} |\psi_{n+6}\rangle + \dots \quad (34)$$

where $f_{\lambda,\beta} = \frac{(\lambda+\beta)}{(1+\lambda\beta)}$. In its compact form one can write,

$$\Psi_n^{(k)} = \sum_{k=0} \left[\frac{(\lambda+\beta)}{(1+\lambda\beta)} \right]^k \sqrt{\frac{(n+2k)!}{n!}} |\psi_{n+2k}\rangle_{\omega_1} \quad (35)$$

The normalization condition here can be written as[6]

$$\langle \psi_n | \Psi_n^{(k)} \rangle = 1 \quad (36)$$

So also the eigenvalue relation

$$\langle \psi_n | H | \Psi_n^{(k)} \rangle = E_n = (n + \frac{1}{2}) \quad (37)$$

VI.(a). Zero Energy Correction Method (Case Study for V=0)

Let the coefficient of $(a^+)^2$ is zero [2] i.e.

$$V = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} - 2(\lambda + \beta)] = 0 \quad (38)$$

which leads to

$$\omega = \omega_2 = \frac{(1 - \beta)}{(1 + \lambda)} \quad (39)$$

In this case, we would like to state that ω calculated using similarity transformation [3] remains the same as ω_2 . Now the perturbation term becomes

$$H_N = U \frac{a^2}{4(1 + \lambda\beta)} = \frac{(\lambda + \beta)}{1 + \lambda\beta} a^2 \quad (40)$$

In this case one will notice that

$$\langle \phi_n | H_N | \phi_{n+2} \rangle = U \frac{\sqrt{[(n+1)(n+2)]}}{4(1 + \lambda\beta)} \quad (41)$$

$$\langle \phi_{n+2} | H_N | \phi_n \rangle = 0 \quad (42)$$

Hence we have

$$\epsilon_n^{(1)} = \langle \phi_n | H_N | \phi_n \rangle = 0 \quad (43)$$

$$\epsilon_n^{(2)} = \sum_{k \neq n} \frac{\langle \phi_n | H_N | \phi_k \rangle \langle \phi_k | H_N | \phi_n \rangle}{(\epsilon_n^{(0)} - \epsilon_k^{(0)})} = \frac{\langle \phi_n | H_N | \phi_{n-2} \rangle \langle \phi_{n-2} | H_N | \phi_n \rangle}{(\epsilon_n^{(0)} - \epsilon_{n-2}^{(0)})} = 0 \quad (44)$$

$$\epsilon_n^{(3)} = \sum_{p,q} \frac{\langle \phi_n | H_N | \phi_p \rangle \langle \phi_p | H_N | \phi_q \rangle \langle \phi_q | H_N | \phi_n \rangle}{(\epsilon_n^{(0)} - \epsilon_p^{(0)})(\epsilon_n^{(0)} - \epsilon_q^{(0)})} = 0 \quad (45)$$

or

$$\epsilon_n^{(3)} = \frac{\langle \phi_n | H_N | \phi_{n-2} \rangle \langle \phi_{n-2} | H_N | \phi_{n-4} \rangle \langle \phi_{n-4} | H_N | \phi_n \rangle}{(\epsilon_n^{(0)} - \epsilon_{n-2}^{(0)})(\epsilon_n^{(0)} - \epsilon_{n-4}^{(0)})} = 0 \quad (46)$$

Here second order correction is zero due to $\langle \phi_n | H_N | \phi_{n-2} \rangle = \delta_{n,n-2}$ and third order correction is zero due to $\langle \phi_{n-4} | H_N | \phi_n \rangle = \delta_{n-2,n-4}$. Similarly one can notice all correction terms $\epsilon_n^{(m)}$ will be zero. Hence

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = (n + \frac{1}{2}) \quad (47)$$

which is the same as the energy level of harmonic oscillator as given in Eq(2) and

$$|\phi_n \rangle = \left(\frac{\sqrt{\omega_2}}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} H_n(\sqrt{\omega_2} x) e^{-\omega_2 \frac{x^2}{2}} \quad (48)$$

VI.(b). Corresponding Wavefunction using Perturbation Theory

Here we consider the wavefunction as

$$\Phi_n^{(k)} = |\phi_n \rangle + f_{\lambda,\beta} \frac{\sqrt{n!}}{2\sqrt{(n-2)!}} |\phi_{n-2} \rangle + (f_{\lambda,\beta})^2 \frac{\sqrt{n!}}{8\sqrt{(n-4)!}} |\phi_{n-4} \rangle + (f_{\lambda,\beta})^3 \frac{\sqrt{n!}}{48\sqrt{(n-6)!}} |\phi_{n-6} \rangle + \dots \quad (49)$$

In its compact form, one can write

$$\Phi_n^{(k)} = \sum_{k=0} \left(\frac{(\lambda + \beta)}{(1 + \lambda\beta)} \right)^k \frac{\sqrt{n!}}{\sqrt{(n-2k)!2^k k!}} |\phi_{n-2k} \rangle_{\omega_2} \quad (50)$$

Here we notice that for $x \rightarrow \infty$ i.e.

$$\phi_n(x \rightarrow \infty) \rightarrow 0 \quad (51)$$

and

$$\Phi_n^{(k)}(x \rightarrow \infty) \rightarrow 0 \quad (52)$$

In this case, the normalization condition can be written as[6]

$$\langle \phi_n | \Phi_n^{(k)} \rangle = 1 \quad (53)$$

So also eigenvalue relation

$$\langle \phi_n | H | \Phi_n^{(k)} \rangle = E_n = \left(n + \frac{1}{2} \right) \quad (54)$$

VII. Comparision with Similarity Transformation using Lie-algebra [4].

In the above we notice that two different frequency(w_1, w_2)corresponds to same energy eigenvalue .Now we compare our results with that of Zhang et.al [4] using Lie-algebra as follows.Previous authors consider the Hamiltonian

$$H = s_0(a^+a + \frac{1}{2}) + s_1(a^+)^2 + s_2a^2 + s_3a^+ + s_4a \quad (55)$$

having energy eigenvalue

$$\epsilon_n = \sqrt{s_0^2 - 4s_1s_2} \left(n + \frac{1}{2} \right) + \frac{s_2s_3^2 + s_1s_4^2 - s_0s_3s_4}{s_0^2 - 4s_0s_1s_2} \quad (56)$$

Case-I $w = w_1 = \frac{1+\beta}{1-\lambda}$.

In this case we have the Hamiltonian

$$H = (a^+a + \frac{1}{2}) - \frac{(\lambda + \beta)}{2(1 + \lambda\beta)}(a^+)^2 \quad (57)$$

Now comparing we get

$$s_0 = 1 \quad (58)$$

$$s_1 = -\frac{(\lambda + \beta)}{2(1 + \lambda\beta)} \quad (59)$$

$$s_2 = 0 \quad (60)$$

$$s_3 = 0 \quad (61)$$

$$s_4 = 0 \quad (62)$$

Hence the

$$\epsilon_n = (n + \frac{1}{2}) \quad (63)$$

which is same as given in Eq(31).

Case-II $w = w_2 = \frac{1-\beta}{1+\lambda}$.

In this case we have the Hamiltonian

$$H = (a^+a + \frac{1}{2}) + \frac{(\lambda + \beta)}{2(1 + \lambda\beta)}a^2 \quad (64)$$

Now comparing we get

$$s_0 = 1 \quad (65)$$

$$s_2 = \frac{(\lambda + \beta)}{2(1 + \lambda\beta)} \quad (66)$$

$$s_1 = 0 \quad (67)$$

$$s_3 = 0 \quad (68)$$

$$s_4 = 0 \quad (69)$$

Hence the

$$\epsilon_n = (n + \frac{1}{2}) \quad (70)$$

which is same as given in Eq(46). Now we see results of Lie-algebra matches with that of perturbation theory on energy level calculation.

VIII. Comparision with Similarity Transformation [3]

It is worth to mention that Fernandez[3] has calculated groundstate wave function of this oscillator using similarity transformation. The explicit expression for wave function [3] is

$$|\phi_0\rangle = \left(\frac{\omega_2}{\pi}\right)^{\frac{1}{4}} e^{-\omega_2 \frac{x^2}{2}} \quad (71)$$

It is easy check that our result in Eq. (45) with $n = 0$ remains the same as that of Fernandez [3]. However, we do not have literature for further comparison.

IX. Conclusion

In this paper, we suggest a simpler procedure for calculating

energy levels and wave function of the non-Hermitian harmonic oscillator under simultaneous transformation of co-ordinate and momentum using perturbation theory . Further energy eigenvalue calculated using perturbation theory matches nicely with that of Lie algebraic method [4]. At this point it is necessary to mention that the ground state wave function calculated by Fernandez [3] refers to zeroth order wave function as reflected in VI(b). However we present a complete picture on wave function. In all the cases, we show that the energy levels remain the same as that of simple Harmonic oscillator. Here we would like to state that if the parameter ω is determined using variational principle [9] i.e. $\frac{d\epsilon_n^{(0)}}{d\omega} = 0$, then one has to calculate all orders of perturbation corrections i.e. $\sum_{m=2}^k \epsilon_n^{(m)}$ because $\langle n|H_N|n+2\rangle \neq 0$ so also $\langle n+2|H_N|n\rangle \neq 0$ and it will be a cumbersome process. However, if the parameter is determined either using $\langle n|H_N|n+2\rangle$ or $\langle n+2|H_N|n\rangle$ as done in above procedure then the energy levels and wave function are obtained easily. Further we notice that

$$\langle \phi_n|H_D|\phi_n\rangle = \langle \psi_n|H_D|\psi_n\rangle = \langle \phi_n|H|\Phi_n^{(k)}\rangle = \langle \psi_n|H|\Psi_n^{(k)}\rangle = \left(n + \frac{1}{2}\right) \quad (72)$$

This directly reflects that non-commuting operators i.e. $[H, H_D] \neq 0$ corresponds

to different wave functions [1]. Lastly, one can use nonlinear perturbation series [10] and easily get convinced that the parameter ω determined using the condition $\langle n|H_N|n+2 \rangle$ or $\langle n+2|H_N|n \rangle$, the calculations become very easy to get the desired result.

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